Discrete 2 Review

Counting Permutations (using $P(n,r)$)

- order of the items is important
- items are individual (ie all different)

Counting Combinations (using $\binom{n}{r} = C(n,r)$)

- counting sets of items
- order is not important
- items are individual (ie all different)

Counting using ** || use combinations also (places to put bars among total slots)

- Counting sets of items
- order is not important
- items are indistinguishable (can have more than one of the same kind)

Soccer example.

There are 18 players on a team

- They want to take a picture of 7 soccer players in a line. How many ways can they do that?

order matters – 18 choices

Ans $= P(18,7) = 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$

$= \frac{18!}{11!}$

- How many different ways can the coach choose 4 defenders from the team?

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order does not matter
18 choices

Ans: $C = \binom{18}{4} = \frac{18 \cdot 17 \cdot 16 \cdot 15}{4 \cdot 3 \cdot 2 \cdot 1}$

- For practice, the coach wants to use 20 soccer balls. There are 3 colors – blue, purple, and red. How many different ways can the balls be chosen?

Blue	Purple	Red
$\times \times \times$	$\times \times$	$\times$

20 stars to place, 2 bars to separate the

kinds of items.

22 slots (for stars or bars)

Need to choose 2 slots for bars.

<u>Ans:</u> $\binom{22}{2} = \frac{22\cdot 21}{2\cdot 1} = .$

<u>Another question:</u> There are 6 different kinds of desserts that we can choose from, and how different ways can we pick 5 desserts to eat next week?

Order does not matter.

with replacement (not ordered)

Desserts #1 | #2 | #3 | #4 | #5 | #6

choose $*$ ← 5 place.

<u># ways:</u> 10 slots → need to choose 5 of them for $*$s.

<u>Ans:</u> $\binom{10}{5}$

<u>Variant</u>: There are 6 different kinds of desserts. How many different ways can we choose one for each work day next week (MTWRF)?

<u>Order matters</u>

$\overline{M} \quad \overline{T} \quad \overline{W} \quad \overline{R} \quad \overline{F}$

6 choices for each.

<u>Multiplication Rule</u>

$6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5$ ways.

(54) $f: A \to B$, $|A| = 100$, $|B| = 12$

(a) Prove $\exists b \in B$ s.t. $|f^{-1}(\{b\})| \geq 9$.

<u>Gen. Pigeonhole</u>: $\begin{pmatrix} 100 \\ \text{things} \\ \text{balls} \end{pmatrix} \longrightarrow \begin{pmatrix} 12 \\ \text{slots} \\ \text{boxes} \end{pmatrix}$

Each $f^{-1}(\{b\})$ is a subset of A,
And they are all disjoint.

$A = \bigcup_{b \in B} f^{-1}(\{b\})$. By the Gen. Pigeonhole Principle,
$\exists$ one box (ie one $b \in B$) that has at

least $\left\lceil \frac{100}{12} \right\rceil$ balls $\Rightarrow \exists\, b \in B$ s.t.

$$\left| f^{-1}(\{b\}) \right| = 9. \blacksquare$$

(b) Is it true that $\forall b \in B$, $\left| f^{-1}(\{b\}) \right| \geq 8$.

Ans: No. For example, if f maps all of A to one element b_0 of B, then

$\left| f^{-1}(\{b_0\}) \right| = 100$, and all other $b \neq b_0$

have $\left| f^{-1}(\{b\}) \right| = 0$. $\blacksquare$

(c) If f is onto, show that No $b \in B$ satisfies $\left| f^{-1}(\{b\}) \right| > 89$.

Proof: Since f is onto $\left| f^{-1}(\{b\}) \right| \geq 1$ for all $b \in B$. If for some $b_0 \in B$, $\left| f^{-1}(\{b_0\}) \right| \geq 90$, then

$$|A| = \sum_{b \in B} \left| f^{-1}(\{b\}) \right| \geq 90 + 11 \cdot 1$$

$= 101$, a contradiction.

$\therefore \left| f^{-1}(\{b\}) \right| < 90 \ \forall b \in B.$ $\blacksquare$

(68) Prove $\binom{c}{b}\binom{b}{a} = \binom{c}{a}\binom{c-a}{b-a}$: $a \le b \le c$

(b) Using formulas

Pf: $LHS = \binom{c}{b}\binom{b}{a} = \frac{c!}{\cancel{b!}(c-b)!}\,\frac{\cancel{b!}}{a!(b-a)!}$

$= \frac{c!}{a!(b-a)!(c-b)!}$.

$RHS = \binom{c}{a}\binom{c-a}{b-a} = \frac{c!}{a!\cancel{(c-a)!}}\,\frac{\cancel{(c-a)!}}{(b-a)!(c-b)!}$

$(c-a)-(b-a) = c-b$

$= \frac{c!}{a!(b-a)!(c-b)!} = LHS.$ ▨

(a) Using a combinatorial argument

$\binom{c}{b}\binom{b}{a} = \binom{c}{a}\binom{c-a}{b-a}$:

Pf: LHS = # of ways to choose b objects from a set of c objects, and then choosing a objects from the set of b chosen.

= # of ways to choose a objects from c objects and then choosing (b-a) objects from the remaining items that comprised the set of b objects

= RHS. ▨

(72) too hard for test.

Variant: Let (F_n) be the Fibonacci sequence, i.e. $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$.

Let $R_n = \frac{F_{n+1}}{F_n}$ be the ratios of successive elements of the Fibonacci sequence. Find a recursive formula for R_n and what it converges to.

$$F_{n+1} = F_n + F_{n-1}$$

divide by F_n $\Rightarrow$ $\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n}$

$$\Rightarrow R_n = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n}$$

$$\Rightarrow R_n = 1 + \frac{1}{F_n/F_{n-1}}$$

R_{n-1}

$$\Rightarrow R_n = 1 + \frac{1}{R_{n-1}}$$

$$R_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1$$

$$\Rightarrow \lim_{n\to\infty} R_n = 1 + \frac{1}{\lim_{n\to\infty} R_{n-1}}$$

Let $L = \lim_{n\to\infty} R_n$

$R_1 = 1$
$R_2 = 1 + \frac{1}{1} = 2$
$R_3 = 1 + \frac{1}{2} = \frac{3}{2}$
$R_4 = \frac{5}{3} \ldots$

$\Rightarrow L = 1 + \frac{1}{L}$

$\Rightarrow L^2 = L + 1$

$\Rightarrow L^2 - L - 1 = 0$

$L = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \geq 0$

$\Rightarrow \boxed{L = \frac{1+\sqrt{5}}{2}}$.

73h Solve (a_n) where

$a_{n+1} = -2a_n - a_{n-1}$, $a_1 = 1$, $a_2 = 1$

Try $a_n = r^n$

$\Rightarrow r^{n+1} = -2r^n - r^{n-1}$

$\Rightarrow r^{n+1} + 2r^n + r^{n-1} = 0$

divide by r^{n-1}: $r^2 + 2r + 1 = 0$

$\Rightarrow (r+1)^2 = 0$

$r_1 = -1$

$\Rightarrow a_n = C_1 (-1)^n + C_2 n (-1)^n$

gen solution

Initial conditions $a_1 = 1 = c_1(-1)^1 + c_2(1)(-1)^1$

$$\Rightarrow 1 = -c_1 - c_2$$

$$a_2 = 1 = c_1(-1)^2 + c_2(2)(-1)^2$$

$$1 = c_1 + 2c_2$$

Add $2 = c_2$

$$1 = -c_1 - 2$$

$$\Rightarrow 3 = -c_1 \Rightarrow c_1 = -3$$

$$\Rightarrow a_n = -3(-1)^n + 2n(-1)^n.$$

Check: $n=1$: $a_1 = (-3)(-1) + 2\cdot 1\cdot(-1)$

$$= 3 - 2 = 1 \checkmark$$

$$a_2 = (-3)(-1)^2 + 2(2)(-1)^2$$

$$= -3 + 4 = 1 \checkmark$$

$$a_3 = (-3)(-1)^3 + 2(3)(-1)^3$$

$$= 3 + -6 = -3$$

$$a_3 = -2a_2 - a_1 = -2(1) - 1 = -3. \checkmark$$